Algebraic solitary-wave solutions of a nonlinear Schrödinger equation

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We present algebraic solitary-wave solutions of a nonlinear Schrödinger equation that includes two terms with power-law nonlinearity. Both bright and dark types are found. Numerical stability analyses show that the solitary wave exhibits a solitonic feature.

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I. INTRODUCTION

It is now well known that many phenomena in contemporary physics can be explained in terms of solitons [1] and, in certain cases, instantons [2]. Of a variety of model equations that predict the existence of solitonic solutions, along with the Korteweg-de Vries (KdV) and the sine-Gordon equations, the type that can be described by a family of nonlinear Schrödinger equations (NLSE's) will be representative. With respect to the topological structure, the transverse configuration, and the asymptotic behavior in the far field, several classification methods of solitonlike fields are possible. For instance, the second classification consists of three categories: a bright, dark, and kink (shock-wave) type. The dark (the topological) type is classifiable further into a gray, black, and darker-than-black solution [3]. A classification is also possible in terms of whether or not the field profile is describable with a combination of exponential functions, such as a hyperbolic function. One may find that the majority of solitary waves that have been found so far are attributable to this family. An exception will be seen in what we call algebraic (or rational) solitons. As the term indicates, the field distribution of the algebraic solitons is expressed by a rational function such as a Lorentzian [4] and, thus, they are localized more weakly than the familiar hyperbolic-type solitons. To date, the Lorentzian solitons (quasisolitons) have been found to exist in some physical contexts. Initially, Zabusky presented this type of soliton as a particular solution of a modified KdV equation with a high-order nonlinear term [5]. Subsequently, Ono attempted to generalize this solution. Furthermore, as a Lorentzian-shaped particular solution of the Benjamin-Ono equation [6,7], Ono derived the similar solution. Numerical experiments by Meiss and Pereira verified the solitonic feature of the solitary wave [8]. In the context of gap solitons, Mills presented a Lorentzian-shaped solitonlike solution, termed a type-II gap soliton, which could be sustained in nonlinear periodic structures within a frequency range near the lower bound of the gap [9]. In a similar context, Grimshaw and Malomed recently showed that at a certain value of the wave velocity, gap solitons in a coupled KdV wave system degenerate into algebraic solitons with a Lorentzian intensity profile [10]. In the study of self-induced transparency of intensified electromagnetic radiation in a three-level medium, it was pointed out by Belenov and Poluéktov that a radiation pulse with the Lorentzian-shape intensity could propagate undistorted in the presence of two-photon resonances [11]. Hanamura attempted to modify their theory and arrived at the conclusion that in the same situation only the semistable Lorentzian pulse is possible [12]. Aside from ordinary solitons mentioned above, Lorentzian-shaped quasi-particle-like fields termed instantons or Euclidean gauge solitons were discovered by Polyakov and t' Hooft in the context of quantum field theory [13]. Aside from their close relevance to the quantum mechanical tunneling and the quantum chromodynamics, a classical interpretation of the instantons may be adoptable as a four-dimensional "static" solitonlike entity that is localized upon a limited volume in the space time [2]. We predict in this paper the existence of a new algebraic solitonic solution through discovery of a particular solution of a NLSE that includes two power-law nonlinear terms. Both bright (nontopological) and dark (topological) types are presented. For the latter, a unique algebraic solitary-wave field with *non-Lorentzian* shape is predicted. Numerical stability analyses show that the solitary wave exhibits a solitonic feature.

II. ALGEBRAIC SOLITARY-WAVE SOLUTIONS

First we consider a generic version of a NLSE that includes two power-law nonlinear terms

$$iu_t + u_{xx} - \gamma_1 |u|^p u + \gamma_2 |u|^{2p} u = 0 , \tag{1}$$

where u is the complex field amplitude that depends on x and t [$u = u(x,t)$], γ_j ($j=1,2$) is a nonvanishing real constant corresponding to the coefficient of $|u|^{jp}u$, and p is a natural number that indicates the order of nonlinearity. We know that Eq. (1), especially that with lower p values, appears in various branches of contemporary physics [1,2]. For instance, in nonlinear optics the case of $p=2$ appears most important since it can model a non-Kerr-type nonlinearity with saturation [14]. For this p value, a bright-type solution that is expressed with a hyperbolic (i.e., a nonalgebraic) function was already presented in the literature [1,14]. For $p=1$, in the context of solitary-wave polaritons, both bright and dark solutions of the hyperbolic type were recently given by us [15]. In these cases the "time" variable t in Eq. (1) should be interpreted as a longitudinal (a wave propagation) axis. The other variable x is then either a retarded time (soliton pulses) or a transverse spatial axis (spatial solitons).

Through a heuristic manner, we have found that solely for $p=1$ and $p=2$, Eq. (1) admits of algebraic solitary-wave solutions: When $p=1$ (quadratic-cubic nonlinearity), for a bright-type algebraic solitary-wave solution we have obtained

$$u(x,t)=u_0L(x;\alpha,1)\ , \tag{2a}$$

with

$$u_0=6\alpha/\gamma_1\ ,\quad \alpha=\tfrac{2}{9}(\gamma_1^2/\gamma_2)\ , \tag{2b}$$

while for a dark-type counterpart, we have derived

$$u(x,t)=u_0|L(x;\alpha,1)-\tfrac{3}{4}|\exp(i\beta t)\ , \tag{3a}$$

with

$$u_0=12\alpha/\gamma_1\ ,\quad \beta=-9\alpha/2\ ,\quad \alpha=\tfrac{1}{18}(\gamma_1^2/\gamma_2)\ . \tag{3b}$$

In the case of $p=2$ (cubic-quintic nonlinearity), we have obtained a bright-type algebraic solitary-wave solution

$$u(x,t)=u_0L(x;\alpha,\tfrac{1}{2})\ , \tag{4a}$$

with

$$u_0=\pm(2\alpha/\gamma_1)^{1/2}\ ,\quad \alpha=\tfrac{3}{4}(\gamma_1^2/\gamma_2)\ , \tag{4b}$$

as well as a dark-type solution

$$u(x,t)=u_0xL(x;\alpha,\tfrac{1}{2})\exp(i\beta t)\ , \tag{5a}$$

with

$$u_0=\pm(6/\gamma_1)^{1/2}\alpha\ ,\quad \beta=-3\alpha\ ,\quad \alpha=\tfrac{1}{12}(\gamma_1^2/\gamma_2)\ . \tag{5b}$$

In Eqs. (2)–(5) the symbol $L(x;\alpha,q)$ indicates an extended version of the Lorentzian, which is defined by

$$L(x;\alpha,q)\equiv(\alpha x^2+1)^{-q}\quad \text{for } \alpha>0,\ q>0\ , \tag{6}$$

where $2\alpha^{-1/2}$ estimates a full width (being a measure of localization) of the intensity profile, and the parameter q outlines a detail of the profile. The distribution function becomes gradual with decreasing q. With $q=1$, Eq. (6) coincides with the exact Lorentzian (super Lorentzian for $q>1$, sub-Lorentzian for $q<1$). The solitary-wave solutions of Eqs. (2a) and (4a) are bright (nontopological), while those given by Eqs. (3a) and (5a) are dark (topological). It should be emphasized here that the topological structure as well as the parity of the two dark fields are radically different. The former [Eq. (3a)] exhibits even symmetry across the center ($x=0$), and the profile varies abruptly across $x=\pm(3\alpha)^{-1/2}$. In contrast to this, like the dark soliton of the cubic NLSE, the latter [Eq. (5a)] is antisymmetric across the center, which results in the total phase shift $\phi_{tot}=\pi$ (black type). We would like to stress here that the dark-type algebraic solitary-wave solution may be unique to the NLSE of Eq. (1).

III. NUMERICAL SIMULATIONS

To evidence the solitonic feature of the present algebraic solitary waves, numerical simulations have been performed with a computational tool previously developed by us [16]. Example results are shown in Figs. 1 and 2 for $p=1$ and $p=2$, respectively. In all cases, the solitary wave being input at $t=0$ is stable and remains unchanged even after the propagation over sufficiently long "times" that attain ten soliton units. Through a Wick rotation, $t\rightarrow -it'$, the paraxial wave equation, Eq. (1), can be translated formally into a reaction-diffusion (an extended Ginzburg-Landau) equation

$$-v_{t'}+v_{xx}-\gamma_1|v|^pv+\gamma_2|v|^{2p}v-\beta v=0\ , \tag{7}$$

where $v(x,t')$ stands for the slowly varying part of $u(x,t')$. Indeed this type of model equation will appear in various contexts of condensed matter physics and of cross-disciplinary physics such as chemical kinetics far from equilibrium, biophysics, and mathematical biology [17]. Numerical experiments with input of the holelike fields [Eqs. (3) and (5)] at $t'=0$ have shown the same results as Figs. 1(b) and 2(b), respectively. However, in

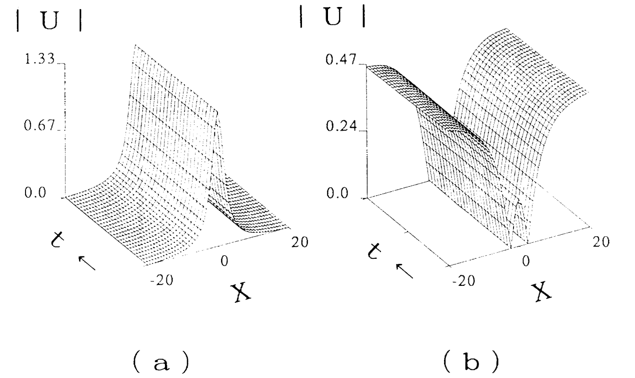

FIG. 1. Propagation of algebraic solitary wave ($p=1$) along the longitudinal (the t) axis. (a) Bright type and (b) dark type. In both types the parameters in Eq. (1) are set to be $\gamma_1=\gamma_2=1$. The solitary-wave field of (a) Eqs. (2) and (b) Eqs. (3) is launched at $t=0$. The total propagation "time" attains 5π, which coincides with ten soliton units, where one soliton unit is taken to be $\pi/2$.

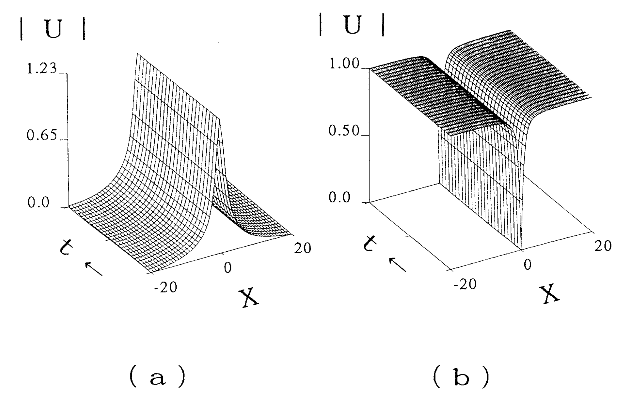

FIG. 2. Same as Fig. 1, but $p=2$. (a) Bright type and (b) dark type. The parameters in Eq. (1) are set to be (a) $\gamma_1=\gamma_2=1$ and (b) $\gamma_1=6$, $\gamma_2=3$. The solitary-wave field of (a) Eqs. (4) and (b) Eqs. (5) is launched at $t=0$.

contrast to such stability of the holes, as might be expected from the Ginzburg-Landau theory, the results for the droplike excitations [Eqs. (2) and (4)] have exhibited a strong decay.

IV. COMPARISON WITH NONALGEBRAIC SOLITARY-WAVE SOLUTIONS

To elucidate the unique features of the algebraic solitary-wave solutions presented in Sec. II, in this section we review the corresponding nonalgebraic solitary-wave solutions for the NLSE [Eq. (1)], which are exponentially localized. In the context of quantum field theory, nonalgebraic bright solitonlike solutions of Eq. (1) were presented for $p=1$ ($\phi^3-\phi^4$ theory) [2] and for $p=2$ ($\phi^4-\phi^6$ theory) [1].

For $p=1$, as a special solution of Eq. (1) we write [2]

$$u(x,t)=u_0[(b^2-4ac)^{1/2}\cosh(a^{1/2}x)+b]^{-1/2}\exp(i\beta t)\ , \tag{8a}$$

with

$$u_0=2a,\ \ \beta=a,\ \ b=-2\gamma_1/3,\ \ c=-\gamma_2/2\ , \tag{8b}$$

where $a>0$ and $b^2>4ac$ (i.e., $u_0>0$, $\beta>0$, and $\gamma_1^2>-9\beta\gamma_2/2$). In Eq. (1), γ_1 may be positive or negative, but to ensure the nonsingularity, for $\gamma_1>0$, γ_2 must also be positive. This is in contrast with the algebraic counterpart [Eqs. (2)] where γ_2 must be positive irrespective of the sign of γ_1. A rather essential difference between the algebraic and the nonalgebraic solutions arises in the limit of $\gamma_2\to 0$. As is obvious from Eq. (2b), in this limit, $|u_0|\to\infty$ and $\alpha^{-1/2}\to 0$, which results in a spiky shape with a δ-functionlike singularity at the center ($x=0$). In sharp contrast to this singularity, for the nonalgebraic solution [Eqs. (8)], as $\gamma_2\to 0$, it approaches the fundamental bright-soliton solution for the quadratic nonlinearity

$$u(x,t)=u_0\mathrm{sech}^2(\alpha x)\exp(i\beta t)\ , \tag{9a}$$

with

$$u_0=-6\alpha^2/\gamma_1\ ,\ \ \beta=4\alpha^2\ . \tag{9b}$$

Here γ_1 may be positive or negative [18]. Note that only recently was the exponentially localized solution of Eqs. (8) discussed in the context of radiation-matter interactions in far-infrared electromagnetic transients [15].

For $p=2$, we obtain [1,14]

$$u(x,t)=u_0[\nu^{1/2}\cosh(2\beta^{1/2}x)+1]^{-1/2}\exp(i\beta t)\ , \tag{10a}$$

with

$$u_0=(-4\beta/\gamma_1)^{1/2}\ ,\ \ \nu=1+\tfrac{16}{3}(\gamma_2/\gamma_1^2)\beta\ , \tag{10b}$$

where $u_0>0$, $\beta>0$, $\nu>0$, $\gamma_1<0$, and $\gamma_2>-(\frac{3}{16})(\gamma_1^2/\beta)$. Note that Eq. (4b) predicts $|u_0|\to\infty$ and $\alpha^{-1/2}\to 0$ as $\gamma_2\to 0$, which takes the form of a spike with the singularity at the center. In sharp contrast to this, for the nonalgebraic solution [Eqs. (10)], as $\gamma_2\to 0$, it approaches the familiar fundamental bright-soliton solution

$$u(x,t)=u_0\mathrm{sech}(\alpha x)\exp(i\beta t)\ , \tag{11a}$$

with

$$u_0=\pm(-2/\gamma_1)^{1/2}\alpha\ ,\ \ \beta=\alpha^2\ . \tag{11b}$$

Note that in the context of nonlinear optics [14], the solution of Eq. (10) was used for studying high-intensity laser-beam (-pulse) propagation in cubic-quintic nonlinear optical media.

V. CONCLUSIONS

We have shown that there exist algebraic solitary-wave solutions for a nonlinear Schrödinger equation. Both bright and dark types have been presented. Numerical stability analyses have shown that the present algebraic solitary waves exhibit, as was expected, a solitonic feature. Their unique features have been elucidated through comparison with the conventional nonalgebraic solitary waves with an exponential tail.

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